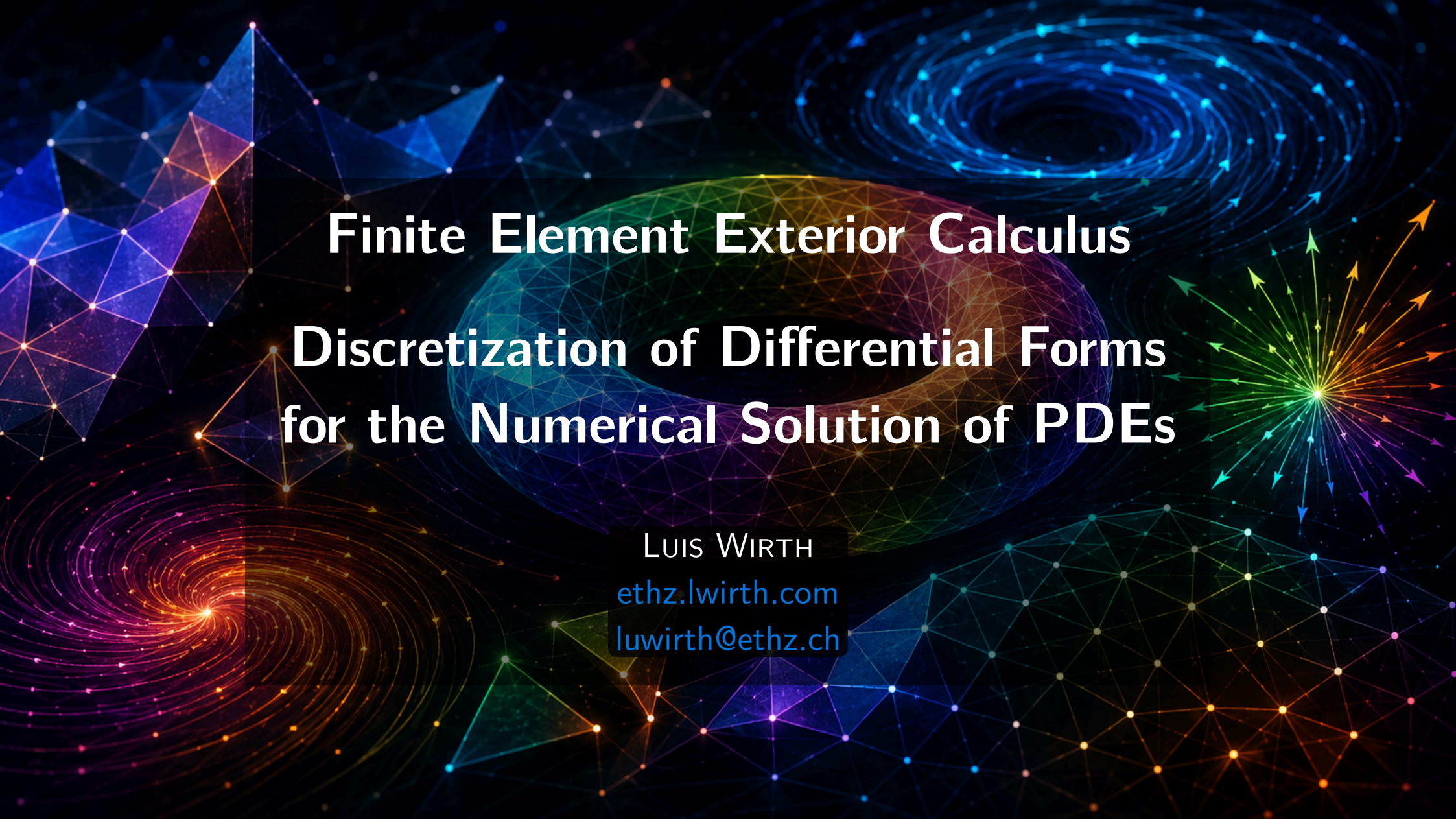


The background is a dark blue space filled with various geometric and mathematical visualizations. On the left, there are several interconnected triangles and polygons in shades of blue and purple, some with glowing points at their vertices. In the center, a large, translucent, multi-colored sphere (with hues of green, yellow, and orange) is composed of a dense network of small triangles. To the right, there are several concentric circles with arrows pointing outwards, suggesting a vector field or a flow. At the bottom right, a cluster of arrows radiates from a central point, with colors transitioning from green to yellow to orange. The overall aesthetic is scientific and mathematical, with a focus on geometry and vector calculus.

Finite Element Exterior Calculus

Discretization of Differential Forms for the Numerical Solution of PDEs

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Respect topology, geometry of the domain $\rightarrow$ obtain correct solutions.

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Big theme: Cohomology

Implementation: Formoniq, a Rust library for FEEC on coordinate-free simplicial complexes in arbitrary dimension (BSc thesis).

The PDE at the heart of FEEC

The Hodge–Laplace equation.

We want to solve

$$\Delta u = f$$

on a Riemannian manifold Ω , where $u, f \in \Lambda^k(\Omega)$ are differential k -forms.

- Generalises scalar ($k = 0$) Laplacian.
- Underlies more sophisticated PDEs like Maxwell's eqs, elasticity eqs, and Stokes flow.

Codifferential $\delta : \Lambda^k \rightarrow \Lambda^{k-1}$

Defined as formal L^2 -adjoint of d (on closed manifolds):

$$\delta := d^* \iff \langle d\omega, \eta \rangle_{L^2} = \langle \omega, \delta\eta \rangle_{L^2}$$

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With boundary we get generalized integration by parts for DF.

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In $\mathbb{R}^3$ (no boundary) these are the adjoint relationships:

$$\mathbf{grad}^* = -\mathbf{div}$$

$$\mathbf{curl}^* = \mathbf{curl}$$

$$\mathbf{div}^* = -\mathbf{grad}$$

Scalar Laplacian

The familiar case, $k = 0$.

On $\Omega \subseteq \mathbb{R}^n$:

$$-\Delta_s u = -\operatorname{div} \operatorname{grad} u$$

In the language of forms:

$$\operatorname{grad}^\flat u = \operatorname{d}u \qquad -\operatorname{div} X = \delta X^\flat$$

So

$$-\Delta_s = -\operatorname{div} \operatorname{grad} \rightsquigarrow \Delta^0 := \delta \operatorname{d}$$

The Hodge–Laplace Operator

Both compositions, permuted.

$$\Delta^k : \Lambda^k(\Omega) \rightarrow \Lambda^k(\Omega), \quad \Delta^k := d\delta + \delta d$$

Symmetric, positive semi-definite, second-order elliptic.

$k = 0 :$	$\Delta^0 = \delta d$	$\rightsquigarrow$	$-\operatorname{div} \operatorname{grad}$
$k \in \{1, \dots, n-1\} :$	$\Delta^k = d\delta + \delta d$	$\rightsquigarrow$	Vector Laplacian
$k = n :$	$\Delta^n = d\delta$	$\rightsquigarrow$	$-\operatorname{div} \operatorname{grad}$

All vector-calculus Laplacians are special cases of one operator.

Hodge-Laplace in $\mathbb{R}^3$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H(\mathbf{grad}) & \xrightarrow{\mathbf{grad}} & H(\mathbf{curl}) & \xrightarrow{\mathbf{curl}} & H(\mathbf{div}) \xrightarrow{\mathbf{div}} L^2 \longrightarrow 0 \\
 \\
 0 & \longleftarrow & L^2 & \xleftarrow{-\mathbf{div}} & \mathring{H}(\mathbf{div}) & \xleftarrow{\mathbf{curl}} & \mathring{H}(\mathbf{curl}) \xleftarrow{-\mathbf{grad}} \mathring{H}(\mathbf{grad}) \longleftarrow 0
 \end{array}$$

k	$\Delta^k = \delta \mathbf{d} + \mathbf{d} \delta$	$\langle \mathbf{d}, \mathbf{d} \rangle + \langle \delta, \delta \rangle$	natural BC	essential BC	Name
0	$-\mathbf{div} \mathbf{grad} + 0$	$\langle \mathbf{grad}, \mathbf{grad} \rangle + 0$	$\partial u / \partial n$	-	Scalar Neumann
1	$\mathbf{curl} \mathbf{curl} - \mathbf{grad} \mathbf{div}$	$\langle \mathbf{curl}, \mathbf{curl} \rangle + \langle \mathbf{div}, \mathbf{div} \rangle$	$\mathbf{curl} \, u \times n$	$u \cdot n$	Vector
2	$-\mathbf{grad} \mathbf{div} + \mathbf{curl} \mathbf{curl}$	$\langle \mathbf{div}, \mathbf{div} \rangle + \langle \mathbf{curl}, \mathbf{curl} \rangle$	$\mathbf{div} \, u$	$u \times n$	Vector
3	$0 - \mathbf{div} \mathbf{grad}$	$0 + \langle \mathbf{grad}, \mathbf{grad} \rangle$	-	u	Scalar Dirichlet

Primal Weak Formulation

Integrate against test forms.

Multiply by $v \in \Lambda^k$ and integrate by parts:

$$\langle (d\delta + \delta d)u, v \rangle = \langle du, dv \rangle + \langle \delta u, \delta v \rangle$$

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We get the following variational equation:

$$u \in H\Lambda^k \cap H^*\Lambda^k : \quad \langle du, dv \rangle_{L^2\Lambda^k} + \langle \delta u, \delta v \rangle_{L^2\Lambda^k} = \langle f, v \rangle_{L^2\Lambda^k} \quad \forall v \in H\Lambda^k \cap H^*\Lambda^k$$

The Sobolev Spaces

Needed for variational formulation

$$H\Lambda^k \cap H^*\Lambda^k$$

$$H\Lambda^k(\Omega) = \{\omega \in L^2\Lambda^k \mid d\omega \in L^2\Lambda^{k+1}\}$$

$$H^*\Lambda^k(\Omega) = \{\omega \in L^2\Lambda^k \mid \delta\omega \in L^2\Lambda^{k-1}\}$$

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In $\mathbb{R}^3$:

$$H\Lambda^0 = H(\mathbf{grad}; \Omega)$$

$$H\Lambda^1 = H(\mathbf{curl}; \Omega)$$

$$H\Lambda^2 = H(\mathbf{div}; \Omega)$$

$$H\Lambda^3 = L^2(\Omega)$$

Complication: Non-Trivial Kernel

Hodge-Laplace is singular.

In general Δ^k has nontrivial kernel $\ker \Delta^k \neq \{0\}$

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In other words our bilinear form $a(u, v)$, is **not coercive** on $H\Lambda^k \cap H^*\Lambda^k$.

It vanishes on the whole kernel.

→ Lax-Milgram fails.

Harmonic Forms

The kernel of Δ^k .

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Hodge's theorem:

$$\mathfrak{H}^k \cong H_{\text{dR}}^k(\Omega)$$

Homology (the study of holes) enters the picture:

Each hole gives rise to a unique harmonic representative.

Number of k -dim holes is Betti number

$$\beta_k = \dim H_{\text{dR}}^k(\Omega)$$

$\Rightarrow$ Topology determines kernel of PDE.

de Rham Cohomology

De Rham Complex

$$0 \rightarrow \Lambda^0(\Omega) \xrightarrow{d} \dots \xrightarrow{d} \Lambda^n(\Omega) \rightarrow 0$$

$$d \circ d = 0$$

Exact ($\omega = d\eta$) $\Rightarrow$ Closed ($d\omega = 0$)

Closed ($d\omega = 0$) $\nRightarrow$ Exact ($\omega = d\eta$)

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De Rham cohomology

$$H_{\text{dR}}^k(\Omega) = \frac{\ker d^k}{\text{im } d^{k-1}} = \frac{\{\omega \mid d\omega = 0\}}{\{\omega \mid \omega = d\eta\}} = \frac{\text{closed}}{\text{exact}}$$

De Rham's theorem

$$H_{\text{dR}}^k(\Omega) \cong (H_k(\Omega))^*$$

Fixing the Kernel Issue

Operator not invertible, but solutions exist **conditionally!**

Fredholm alternative (self-adjoint case):

- Existence of solution iff $f \perp \ker \Delta = \mathfrak{H}^k$.
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Possible fix:

- Uniqueness: quotient out by harmonics
- Existence: project out harmonic part of f

$$u \in H\Lambda^k \cap H^*\Lambda^k / \mathfrak{H}^k \quad \text{with} \quad a(u, v) = \langle f - P_{\mathfrak{H}} f, v \rangle.$$

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But there are actually two better fixes.

Uniqueness: Gauge Condition

Instead of using a quotient space and having to deal with equivalence classes.

We get uniqueness by enforcing a Gauge

$$u \perp \mathfrak{H}^k$$

This picks out a unique representative.

In weak form

$$\langle u, q \rangle = 0 \quad \forall q \in \mathfrak{H}^k$$

Existence: Hodge Decomposition

Every form splits into three L^2 -orthogonal pieces.

$$\Lambda^k(\Omega) = \underbrace{d\Lambda^{k-1}}_{\substack{\text{exact} \\ \omega=d\alpha}} \oplus \underbrace{\delta\Lambda^{k+1}}_{\substack{\text{coexact} \\ \omega=\delta\beta}} \oplus \underbrace{\mathfrak{H}^k}_{\substack{\text{harmonic} \\ \Delta\omega=0}}$$

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we're missing the harmonic part. Let's introduce it!

$$d\delta u + \delta du + p = f \quad \Rightarrow \quad d\delta u + \delta du = f - p$$

Existence!

Well-posed Problem

We arrive at

Find $(u, p) \in (H\Lambda^k \cap H^*\Lambda^k) \times \mathcal{H}^k$, s.t.

$$\langle du, dv \rangle + \langle \delta u, \delta v \rangle + \langle p, v \rangle = \langle f, v \rangle \quad \forall v \in H\Lambda^k \cap H^*\Lambda^k$$

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But:

Hard to find FE spaces conforming to both $H\Lambda^k$ and $H^*\Lambda^k$ simultaneously...

→ We need to get rid of $H^*\Lambda^k$!

How? Eliminate codifferential!

Mixed Weak Formulation

Let's eliminate the codifferential

Introduce $\sigma = \delta u$ as an auxiliary variable.

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We enforce this weakly using the adjoint property.

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$$\langle \sigma, \tau \rangle = \langle \delta u, \tau \rangle \iff \langle \sigma, \tau \rangle = \langle u, d\tau \rangle$$

And then the codifferential bilinear form becomes

$$\langle \delta u, \delta v \rangle = \langle \sigma, \delta v \rangle = \langle d\sigma, v \rangle$$

No more codifferential! Only exterior derivative.

Final Weak Formulation

Built-in Hodge Decomposition

Find $(\sigma, u, p) \in H\Lambda^{k-1} \times H\Lambda^k \times \mathfrak{H}^k$ such that

$$\begin{aligned}\langle \sigma, \tau \rangle - \langle u, d\tau \rangle &= 0 & \forall \tau \in H\Lambda^{k-1} \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle & \forall v \in H\Lambda^k \\ \langle u, q \rangle &= 0 & \forall q \in \mathfrak{H}^k\end{aligned}$$

- Eq. 1: σ is the **weak** codifferential of u .
- Eq. 2: Weak Hodge decomposition of f : exact $d\sigma$ + coexact δdu + harmonic p
- Eq. 3: Gauge constraint $u \perp \mathfrak{H}^k$

Saddle-point system. Well-posed by Brezzi inf-sup (Arnold–Falk–Winther).

Discretization

How do we make all of this finite-dimensional?

Let's start with the manifold. We need a mesh.

Simplices

The atoms of the mesh.

A Euclidean k -simplex is the convex hull of $k + 1$ affinely independent points:

$$\sigma = \text{conv}\{v_0, \dots, v_k\} = \left\{ \sum_{i=0}^k \lambda^i v_i \mid \lambda^i \geq 0, \sum_{i=0}^k \lambda^i = 1 \right\}.$$

- 0-simplex: vertex.
- 1-simplex: edge.
- 2-simplex: triangle.
- 3-simplex: tetrahedron.

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A abstract combinatorial k -simplex is coordinate-free and doesn't need an embedding:

$$\sigma = [v_0, \dots, v_k] \in \mathbb{N}^{k+1}, \quad v_i \in \mathbb{N}.$$

Two orientations: even and odd permutations of vertex order.

Boundary of a Simplex

$$\partial[v_0, \dots, v_k] = \sum_{i=0}^k (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k]$$

Drop one vertex at a time, with alternating sign.

edge: $\partial[v_0, v_1] = [v_1] - [v_0]$

triangle: $\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1] = [v_0, v_1] + [v_1, v_2] + [v_2, v_0]$

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Key property:

$$\partial \circ \partial = 0.$$

Reminds you of exterior derivative $d \circ d = 0$?

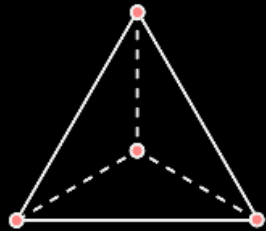
Simplicial Complex

All subsimplicies included.

A simplicial complex $\mathcal{M}$ is a collection of simplices such that

- every subsimplex of a simplex in $\mathcal{M}$ is also in $\mathcal{M}$,
- any two simplices intersect in a common subsimplex.

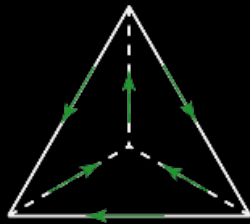
0-simplex (vertices)



$\Delta_0(\mathcal{M})$

$\xleftarrow{\partial}$

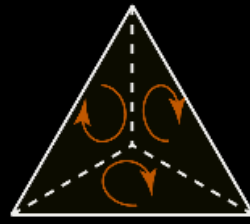
1-simplex (edges)



$\Delta_1(\mathcal{M})$

$\xleftarrow{\partial}$

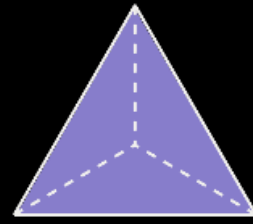
2-simplex (triangles)



$\Delta_2(\mathcal{M})$

$\xleftarrow{\partial}$

3-simplex (tets)



$\Delta_3(\mathcal{M})$

Simplicial Chain Complex

Linear combinations of simplices.

$$C_k(\mathcal{M}) = \left\{ \sum_i c_i \sigma_i \mid \sigma_i \in \Delta_k(\mathcal{M}), c_i \in \mathbb{R} \right\}$$

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This is a complex!

$$0 \xleftarrow{\partial} C_0(\mathcal{M}) \xleftarrow{\partial} C_1(\mathcal{M}) \xleftarrow{\partial} C_2(\mathcal{M}) \xleftarrow{\partial} C_3(\mathcal{M}) \xleftarrow{\partial} 0$$
$$\partial^2 = \partial \circ \partial = 0$$

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Simplicial homology

Boundary ($c = \partial e$) $\Rightarrow$ cycle ($\partial c = 0$)

Cycle ($\partial c = 0$) $\nRightarrow$ boundary ($c = \partial e$). Hole!

$$H_k(\mathcal{M}) = \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}} = \frac{\{c \mid \partial c = 0\}}{\{c \mid c = \partial d\}} = \frac{\text{cycles}}{\text{boundaries}}$$

Computational Boundary Operator

Just a signed incidence matrix.

Order the simplices in each $\Delta_k(\mathcal{M})$.

The signed incidence matrix

$$\mathbf{D}_k \in \{-1, 0, +1\}^{N_{k-1} \times N_k}$$

has entries

$$(\mathbf{D}_k)_{ij} = \begin{cases} +1 & \text{if } \sigma_i \sqsubseteq +\sigma_j \\ -1 & \text{if } \sigma_i \sqsubseteq -\sigma_j \\ 0 & \text{otherwise.} \end{cases}$$

Mesh Geometry

Two routes:

- **Extrinsic:** Embedding $\Omega \hookrightarrow \mathbb{R}^N$, inherit Euclidean geometry (standard FEM)
- **Intrinsic:** Riemannian metric directly on each simplex.

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Intrinsic is better:

- Mesh abstract manifolds (flat torus, Klein bottle) without ambient $\mathbb{R}^N$,
- Extends naturally to Pseudo-Riemannian geometry, e.g.
 - FEEC on 4D spacetimes with Lorentzian metric

Metric on a Simplex

Gram matrix from a basis of the tangent space.

A simplex is intrinsically flat $\Rightarrow$ metric is constant on it.

Globally metric is piecewise constant.

By picking a basis we can represent the metric as a matrix.

$\Rightarrow$ Spanning vectors $e_i = v_i - v_0$

$$\mathbf{G}_{ij} = g(e_i, e_j) \in \mathbb{R}.$$

One Gram matrix per top-level simplex.

Intrinsic Geometry via Edge Lengths

Some Regge calculus

No extrinsic vertex coordinates but intrinsic edge lengths!

Given edge lengths

$$d : \Delta_1(\mathcal{M}) \rightarrow \mathbb{R}^+, \quad d_{ij} = d([v_i, v_j]),$$

the Gram matrix is determined:

$$\mathbf{G}_{ij} = \frac{1}{2}(d_{0i}^2 + d_{0j}^2 - d_{ij}^2)$$

and conversely

$$d_{ij} = \sqrt{G_{ii} + G_{jj} - 2G_{ij}}$$

Mesh finished.

- Topology as simplicial complex.
- Geometry as edge lengths.

Discrete Differential Forms

Defined on the mesh

- **Simplicial Cochains:**
 - Discretized Differential Forms
 - DOF coefficients
 - Discrete Exterior Derivative
- **Whitney Forms:**
 - Reconstructed Continuous Differential Forms
 - FE basis functions

Simplicial Cochains

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These are the actual discrete objects.

There exists a discrete exterior derivative d_h .

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A discrete k -form is a k -cochain: a real-valued function on k -simplices:

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Discretization of continuous forms by the **de Rham map** (integration):

$$\mathcal{I} : \begin{cases} \Lambda^k(\Omega) \rightarrow C^k(\mathcal{M}) \\ \omega \mapsto \omega_h \end{cases} \quad \text{with} \quad \omega_h(\sigma) = \int_{\sigma} \omega.$$

Chains and Cochains

$C_k(\mathcal{M})$ and $C^k(\mathcal{M})$ are **dual** finite-dimensional vector spaces.

Chain C_k — subscript	Cochain C^k — superscript
Linear combination of k -simplices	Linear functional on k -chains
Geometric — what you integrate over	Analytic — what you integrate
Boundary $\partial_k : C_k \rightarrow C_{k-1}$	Differential $d^k : C^k \rightarrow C^{k+1}$
Homology H_k — the holes themselves	Cohomology H^k — dual to holes

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The **duality pairing** represents discrete integration.

$$\langle \cdot, \cdot \rangle : C_k(\mathcal{M}) \times C^k(\mathcal{M}) \rightarrow \mathbb{R}$$
$$\langle \omega, c \rangle = \sum_{(c_i, \sigma_i) \in c} c_i \omega(\sigma_i) = \int_c \omega$$

Discrete Exterior Derivative

d is the dual map of ∂ .

Stokes' theorem links integration and boundary:

$$\int_{\partial c} \omega = \int_c d\omega.$$

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Computationally: transpose of the incidence matrix,

$$d^k = \mathbf{D}_{k+1}^\top.$$

Purely topological. No metric.

Whitney Forms

For Galerkin we need a finite-dimensional function subspace.
These are the spaces of **Whitney forms**.

$$\mathcal{W}\Lambda^k(\Omega) \subseteq H\Lambda^k(\Omega)$$

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Unify standard FE families:

$\mathcal{W}\Lambda^0 = \mathcal{P}_1$	Lagrange (nodal)
$\mathcal{W}\Lambda^1 = \mathcal{N}$	Nédélec edge elements
$\mathcal{W}\Lambda^{n-1} = \mathcal{RT}$	Raviart–Thomas
$\mathcal{W}\Lambda^n = \mathcal{P}_0^{\text{DG}}$	piecewise constant

Whitney Basis

Basis functions for Whitney k -forms live on k -simplices.

$$\mathcal{W}\Lambda^k(\mathcal{M}) = \text{span} \{ \varphi_\sigma : \sigma \in \Delta_k(\mathcal{M}) \}$$

- $\mathcal{W}\Lambda^0(\mathcal{M})$ on 0-simplices $\Delta_0(\mathcal{M})$
- $\mathcal{W}\Lambda^1(\mathcal{M})$ on 1-simplices $\Delta_1(\mathcal{M})$
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Whitney basis property (generalization of Lagrange nodal basis property):

$$\int_\tau \varphi_\sigma = \begin{cases} +1 & \text{if } \tau = +\sigma \\ -1 & \text{if } \tau = -\sigma \\ 0 & \text{otherwise.} \end{cases}$$

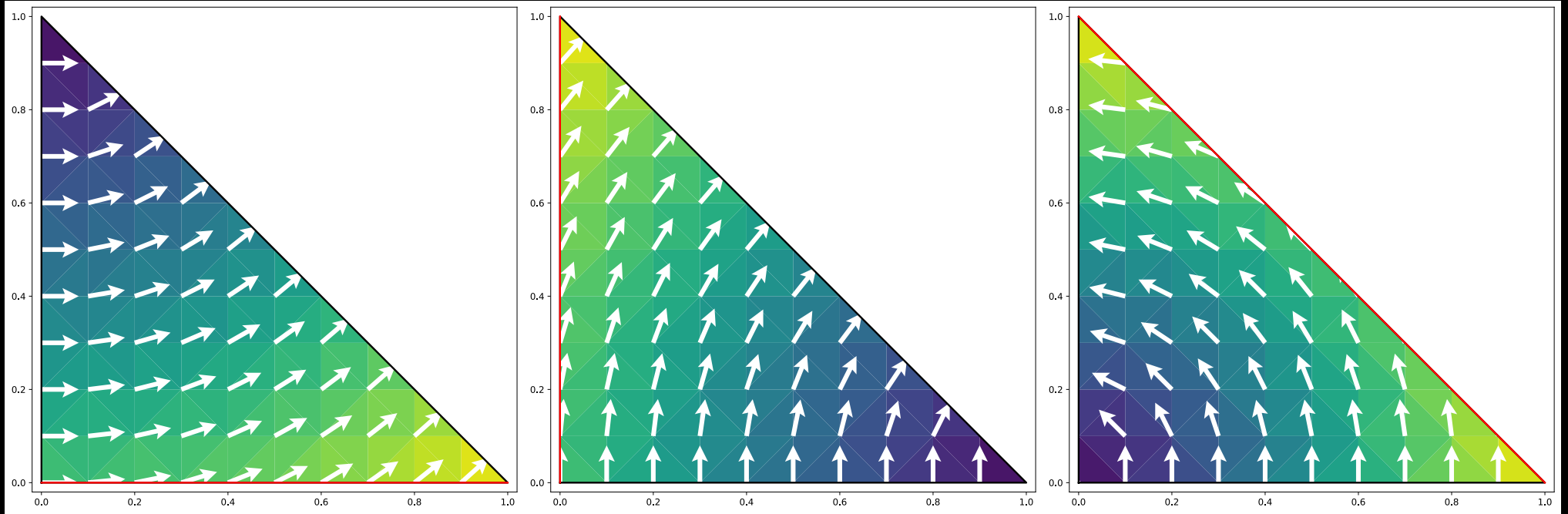
Whitney LSF

Local shape functions defined using barycentric coordinate functions $\lambda_i \in \Lambda^0(K)$, of top-level simplex K . Each associated with a vertex $v_i \in K$.

Whitney k -form of a subsimplex $\sigma = [v_{i_0}, \dots, v_{i_k}] \subseteq K$:

$$\varphi_\sigma|_K = \lambda_{i_0 \dots i_k} = k! \sum_{l=0}^k (-1)^l \lambda_{i_l} \left(d\lambda_{i_0} \wedge \dots \wedge \widehat{d\lambda_{i_l}} \wedge \dots \wedge d\lambda_{i_k} \right)$$

Whitney 1-Form LSF on Reference 2-Simplex

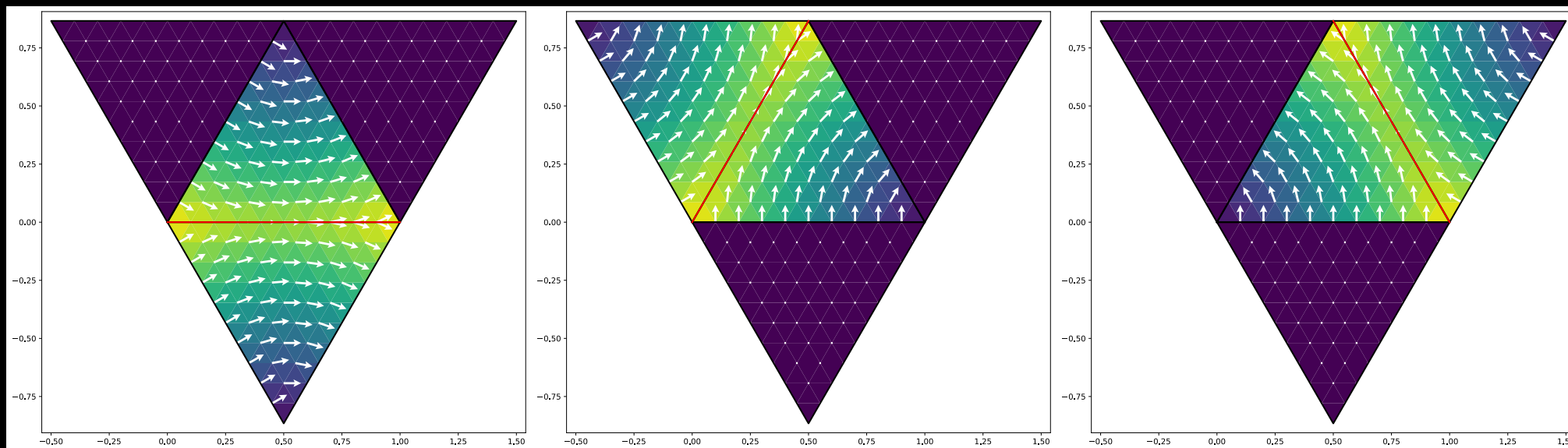


$$\lambda_{01} = (1 - y)dx + xdy$$

$$\lambda_{02} = ydx + (1 - x)dy$$

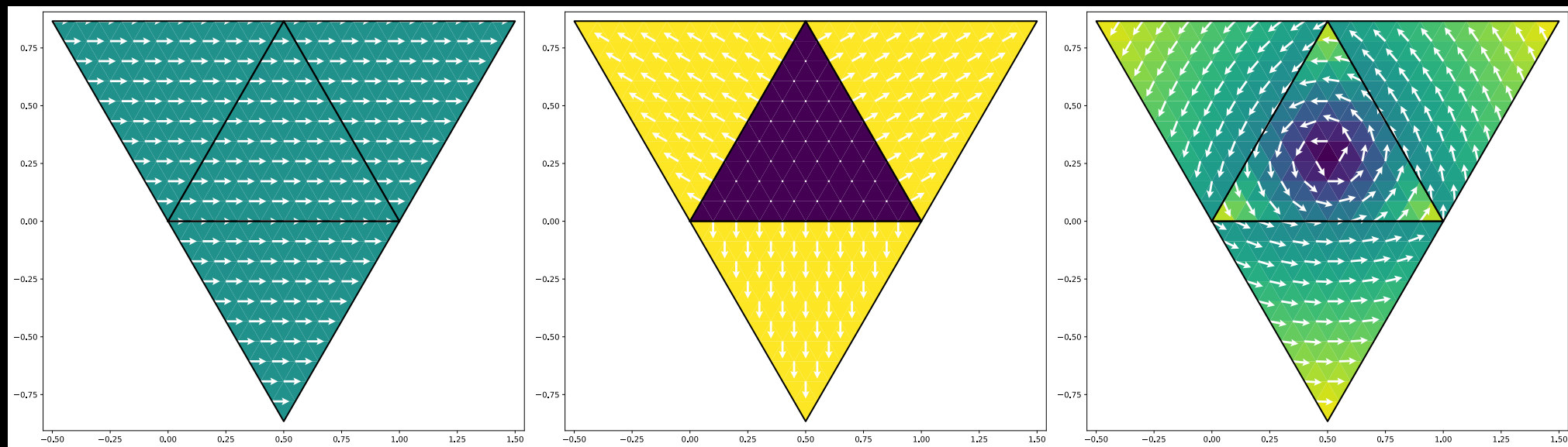
$$\lambda_{12} = -ydx + xdy$$

Whitney 1-Form GSF on Triforce Mesh



Example Whitney 1-Forms on Triforce Mesh

Linear Combination



Whitney Reconstruction

Whitney forms are linear combinations of Whitney basis functions. Coefficients are cochain values (DoFs) on simplices.

$$u = \sum_{\sigma \in \Delta_k(\mathcal{M})} u_\sigma \varphi_\sigma$$

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Reconstruction of discrete cochains by the **Whitney map** (interpolation):

$$\mathcal{W} : C^k(\mathcal{M}) \rightarrow \mathcal{W}\Lambda^k(\Omega)$$

$$\mathcal{W}(\omega) = \sum_{\sigma \in \Delta_k(\mathcal{M})} u_\sigma \varphi_\sigma$$

The de Rham and Whitney Maps

$$\mathcal{I} : \Lambda^k(\Omega) \rightarrow C^k(\mathcal{M}), \quad \mathcal{W} : C^k(\mathcal{M}) \rightarrow \mathcal{W}\Lambda^k \subseteq H\Lambda^k(\Omega).$$

- Integration: $\mathcal{I}(\omega)(\sigma) = \int_{\sigma} \omega$.
- Reconstruction: $\mathcal{W}(c) = \sum_{\sigma} c(\sigma) \varphi_{\sigma}^k$.

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Cochain map property (maps commute with d):

$$\mathcal{I} \circ d = d \circ \mathcal{I}, \quad \mathcal{W} \circ d = d \circ \mathcal{W}.$$

Because $\mathcal{I}$ and $\mathcal{W}$ are cochain maps, they induce isomorphisms on cohomology.

The Structure Preservation

The commutative diagram

$$\begin{array}{ccc} H\Lambda^k(\Omega) & \xrightarrow{\quad d \quad} & H\Lambda^{k+1}(\Omega) \\ \downarrow \mathcal{I} & & \downarrow \mathcal{I} \\ C^k(\mathcal{M}) & \xrightarrow{\quad d \quad} & C^{k+1}(\mathcal{M}) \\ \downarrow \mathcal{W} & & \downarrow \mathcal{W} \\ \mathcal{W}\Lambda^k(\mathcal{M}) & \xrightarrow{\quad d \quad} & \mathcal{W}\Lambda^{k+1}(\mathcal{M}) \end{array}$$

The Whitney Complex

FEEC projection operator onto Whitney subspace

$$\Pi_h := \mathcal{W} \circ \mathcal{I} : H\Lambda^k \rightarrow \mathcal{W}\Lambda^k$$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H\Lambda^0(\Omega) & \xrightarrow{d} & H\Lambda^1(\Omega) & \xrightarrow{d} & \dots \xrightarrow{d} H\Lambda^n(\Omega) \longrightarrow 0 \\
 & & \downarrow \Pi_h & & \downarrow \Pi_h & & \downarrow \Pi_h \\
 0 & \longrightarrow & \mathcal{W}\Lambda^0(\mathcal{M}) & \xrightarrow{d} & \mathcal{W}\Lambda^1(\mathcal{M}) & \xrightarrow{d} & \dots \xrightarrow{d} \mathcal{W}\Lambda^n(\mathcal{M}) \longrightarrow 0
 \end{array}$$

Each square commutes: $\Pi_h \circ d = d \circ \Pi_h$.

$\Rightarrow$ The Whitney complex is a **subcomplex** of the de Rham complex.

$$\mathcal{W}\Lambda^\bullet \subseteq H\Lambda^\bullet$$

Galerkin Mixed Formulation

Replace $H\Lambda^k$ by $\mathcal{W}\Lambda^k(\mathcal{M})$.

Find $(\sigma_h, u_h, p_h) \in \mathcal{W}\Lambda^{k-1} \times \mathcal{W}\Lambda^k \times \mathfrak{H}_h^k$ such that

$$\begin{aligned}\langle \sigma_h, \tau \rangle - \langle u_h, \mathrm{d}\tau \rangle &= 0 & \forall \tau \in \mathcal{W}\Lambda^{k-1} \\ \langle \mathrm{d}\sigma_h, v \rangle + \langle \mathrm{d}u_h, \mathrm{d}v \rangle + \langle p_h, v \rangle &= \langle f, v \rangle & \forall v \in \mathcal{W}\Lambda^k \\ \langle u_h, q \rangle &= 0 & \forall q \in \mathfrak{H}_h^k\end{aligned}$$

Convergence guaranteed by FEEC projection $\Pi_h = \mathcal{W} \circ \mathcal{I}$.

Convergence rate $O(h)$ for first-order Whitney.

Matrix Galerkin Mixed Formulation

$$\sum_j \sigma_j \langle \varphi_j^{k-1}, \varphi_i^{k-1} \rangle - \sum_j u_j \langle \varphi_j^k, d\varphi_i^{k-1} \rangle = 0$$

$$\sum_j \sigma_j \langle d\varphi_j^{k-1}, \varphi_i^k \rangle + \sum_j u_j \langle d\varphi_j^k, d\varphi_i^k \rangle + \sum_j p_j \langle \eta_j^k, \varphi_i^k \rangle = \langle f, \varphi_i^k \rangle$$

$$\sum_j u_j \langle \varphi_j^k, \eta_i^k \rangle = 0$$

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$$\sum_j u_j \langle \varphi_j^k, \eta_i^k \rangle = 0$$

Given $\mathbf{f} \in \mathbb{R}^{N_k}$, find $(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{p}) \in (\mathbb{R}^{N_{k-1}} \times \mathbb{R}^{N_k} \times \mathbb{R}^{N_k})$ s.t.

$$\mathbf{M}^{k-1} \boldsymbol{\sigma} - (\mathbf{d}^{k-1})^\top \mathbf{M} \mathbf{u} = 0$$

$$\mathbf{M} \mathbf{d} \boldsymbol{\sigma} + \mathbf{d}^\top \mathbf{M}^{k+1} \mathbf{d} \mathbf{u} + \mathbf{M} \mathbf{H} \mathbf{p} = \mathbf{f}$$

$$\mathbf{H}^\top \mathbf{M} \mathbf{u} = 0$$

Mass Element Matrix

Galerkin Geometry

Local mass matrix on a top-simplex K :

$$\mathbf{M}_K^k = \left[\langle \varphi_i^k, \varphi_j^k \rangle_{L^2 \Lambda^k(K)} \right]_{i,j}$$

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Expanding the Whitney LSF:

$$\begin{aligned} (M_K^k)_{i,j} &= \langle \lambda_{i_0 \dots i_k}, \lambda_{j_0 \dots j_k} \rangle_{L^2 \Lambda^k(K)} \\ &= k!^2 \sum_{l,m} (-1)^{l+m} \left\langle \begin{matrix} d\lambda_{i_0} \wedge \dots \wedge \widehat{d\lambda}_{i_l} \wedge \dots \wedge d\lambda_{i_k} \\ d\lambda_{j_0} \wedge \dots \wedge \widehat{d\lambda}_{j_m} \wedge \dots \wedge d\lambda_{j_k} \end{matrix} \right\rangle_{\Lambda^k} \int_K \lambda_{i_l} \lambda_{j_m} \text{vol}_g \end{aligned}$$

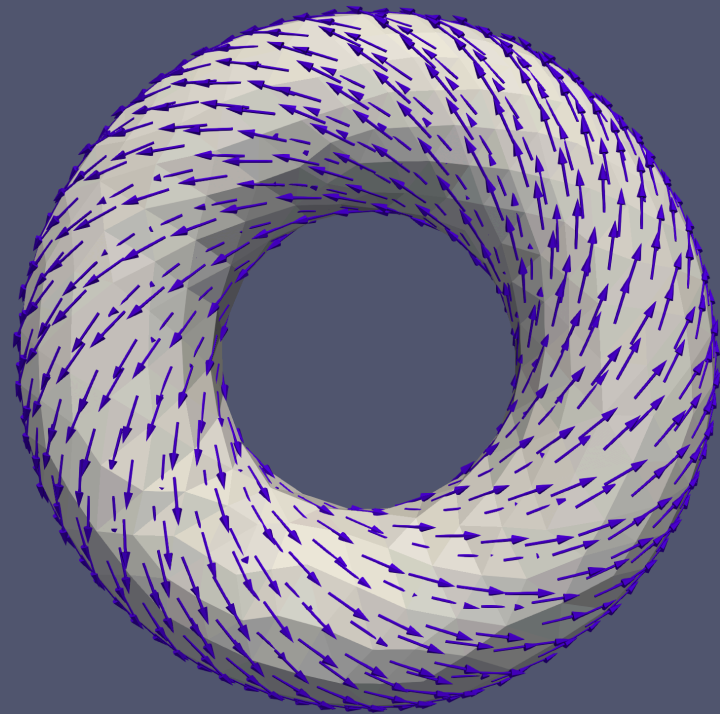
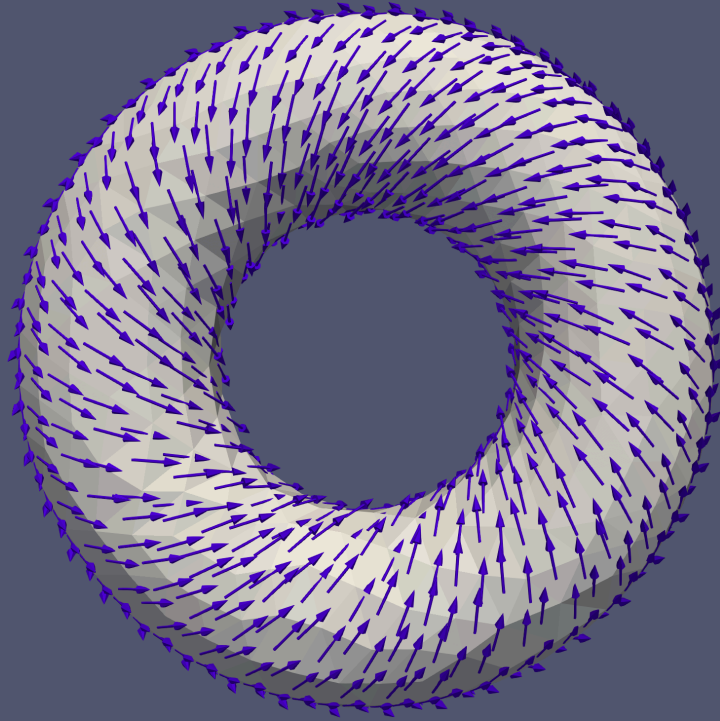
This Completes the Construction

My library can solve

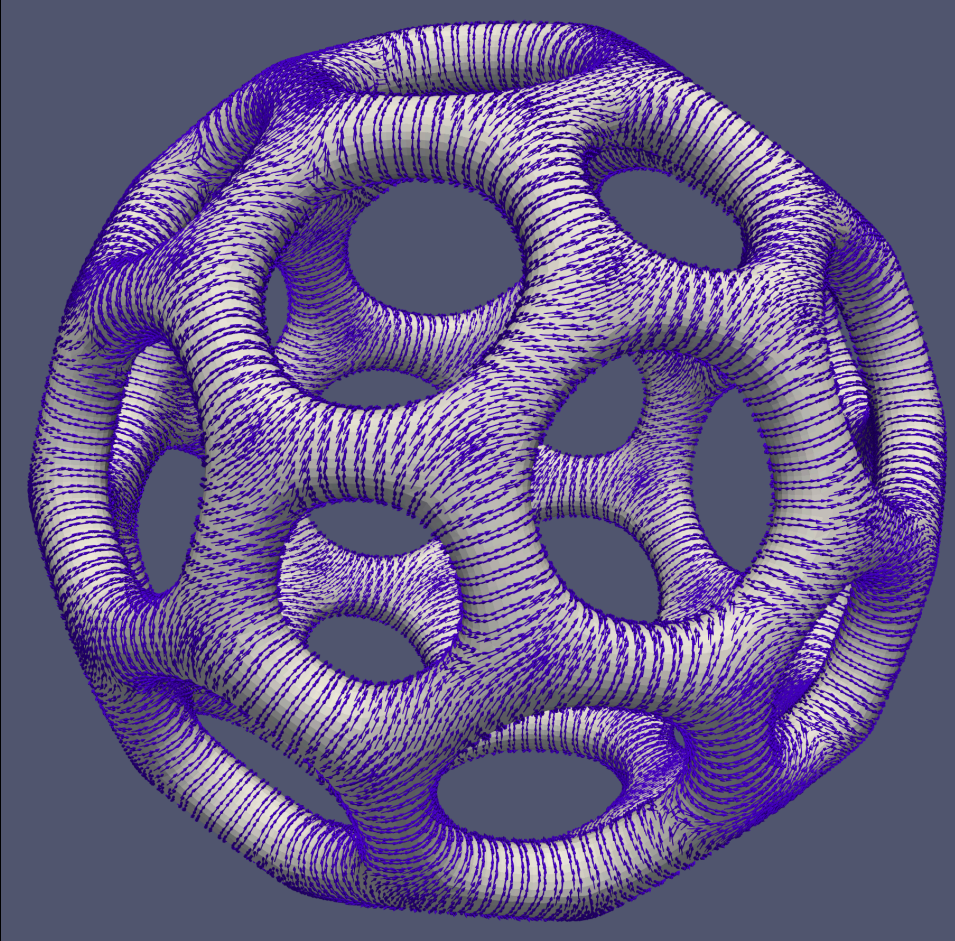
- Hodge-Laplace
 - Eigenvalue problem
 - Source problem
- On any mesh
 - any dimension
 - any topology

Let's look at some results.

Harmonic 1-Forms on $\mathbb{T}^2$, with $\beta_1 = 2$



A Harmonic 1-Forms On a Football Mesh



Summary

FEEC combines three threads:

- Differential geometry: manifolds, differential forms, de Rham.
- Algebraic topology — simplicial complexes, boundaries, homology.
- Functional analysis — Sobolev spaces, variational formulations.

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- Differential geometry: manifolds, differential forms, de Rham.
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Implementation choices in Formoniq:

- Arbitrary dimensional manifolds with non-trivial topology
- Coordinate-free geometry
- Arbitrary form degrees
- 1st order Whitney forms
- Hodge–Laplace source and eigenvalue problems.

Extensions

- The Whitney space is the lowest-order **trimmed polynomial** space $\mathcal{W}\Lambda^k = \mathcal{P}_1^-\Lambda^k$.
 - Extend to higher-degree polynomial FEEC spaces.
- Lorentzian geometry for 4D spacetime.
- Solve other PDEs
 - Maxwell!!

Thank you.

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